Integration by parts and more difficult techniques

1. Evaluate:

(a) $\int \sec^4 x\,dx$ **(b)** $\int \sec^5 x\,dx$ **(c)** $\int \ln x\,dx$ **(d)** $\int x a^x dx$

(e) $\int (e^x + 2x)^2 dx$ **(f)** $\int \frac{\ln x dx}{(1+x)^2}$ **(g)** $\int \frac{\ln(x+1)dx}{\sqrt{x+1}}$ **(h)** $\int e^x \cos x\,dx$

(i) $\int e^{-\frac{x}{2}} \cos 2x\,dx$ **(j)** $\int \sqrt{x}\ln x\,dx$ **(k)** $\int \ln\left(x+\sqrt{1+x^2}\right)dx$ **(l)** $\int \left(\sin^{-1} x\right)^2 dx$

(m) $\int_{1/2}^{1} x \ln\frac{1}{x}\,dx$ **(n)** $\int_{-\pi/4}^{\pi/4} \tan^3 x\,dx$ **(o)** $\int_0^{\pi/4} e^{2x}\sin^2 x\,dx$ **(p)** $\int_0^{1/2} x\ln\frac{1+x}{1-x}dx$

2. Evaluate:

(a) $\int \frac{ds}{\sqrt[3]{2s^2}}$ **(b)** $\int \left(2\sqrt{y} - 5y^{1.2} + 2y^{-0.2} + y^2 + 3\right)dy$ **(c)** $\int \frac{3dz}{(2-3z)^3}$

(d) $\int (2x+5)^4 dx$ **(e)** $\int_0^4 \frac{dt}{2t+1}$ **(f)** $\int \frac{2dp}{1-3p}$ **(g)** $\int 2e^{3x+2}dx$

(h) $\int \frac{dy}{5e^{0.3y}}$ **(i)** $\int 2^{3z}dz$ **(j)** $\int \sin(2-3\theta)d\theta$ **(l)** $\int_0^3 \frac{dt}{9+t^2}$

(m) $\int_0^2 \frac{dx}{16-x^2}$ **(n)** $\int_0^1 \frac{dy}{\sqrt{2y-y^2}}$ **(o)** $\int \frac{dz}{\sqrt{9+4z^2}}$ **(p)** $\int \frac{ds}{\sqrt{9s^2-1}}$

(q) $\int \frac{4t+1}{4t^2+2t-1}dt$ **(r)** $\int \frac{x^2}{(x+1)(x-2)}dx$ **(s)** $\int \frac{2p^2-1}{\sqrt{2p^3-3p+2}}dp$ **(t)** $\int \frac{dz}{z^2+2z+10}$

(u) $\int_0^1 \frac{x^4}{1+x^2}dx$ **(v)** $\int_0^2 \frac{3+2s}{1+2s}ds$ **(w)** $\int \frac{dy}{\sqrt{y^2+2y+10}}$ **(x)** $\int \frac{u+5}{\sqrt{u^2+4u+5}}du$

(y) $\int \frac{p+5}{p^2+4p+5}dp$ **(z)** $\int \frac{x^n}{1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}}dx$

3. Evaluate:

(a) $\int \frac{dx}{x^2\left(x^2+a^2\right)}$ **(b)** $\int_0^2 \frac{dz}{z^3+8}$ **(c)** $\int \frac{y^3}{\left(a^2+y^2\right)^2}dy$ **(d)** $\int \frac{x-3}{\sqrt{4x^2-4x-3}}dx$

(e) $\int \frac{2p+5}{\sqrt{18p-9p^2-5}}dp$ **(f)** $\int \frac{dz}{\left(z^2-1\right)\left(z^2-3z+2\right)}$ **(g)** $\int \frac{du}{u^4+3u^2+2}$ **(h)** $\int \sqrt{10+6x+x^2}\,dx$

(i) $\int \sqrt{4v^2-8v-5}\,dv$ **(j)** $\int_0^{2\pi} \sin 4\theta\,d\theta$ **(k)** $\int_0^{\pi/8} \frac{d\phi}{\cos^2 2\phi}$ **(l)** $\int \sin^2\frac{x}{2}\cos\frac{x}{2}dx$

(m) $\int \frac{du}{3\sin u + 4\cos u}$ **(n)** $\int \frac{dt}{(5\sin t - 12\cos t)^2}$ **(o)** $\int_0^{\pi/2} \frac{d\theta}{4+5\cos^2\theta}$ **(p)** $\int \frac{dv}{4+5\cos v}$

(q) $\int_0^{\pi/3} \sin^2 2\theta\,d\theta$ **(r)** $\int \cos 2n\theta \sin 3n\theta\,d\theta$ **(s)** $\int \cos 2nt \cos 5nt\,dt$ **(t)** $\int \sin 5\phi \sin 7\phi\,d\phi$

(u) $\int_0^{\pi/2} \sin^5 x\, dx$ **(v)** $\int_0^{\pi/2} \sin^3 x \cos^4 x\, dx$ **(w)** $\int \sin^6 x\, dx$ **(x)** $\int \sin^2\theta \cos^4\theta\, d\theta$

4. Evaluate:

(a) $\int_{-\pi/2}^{\pi/2} \sin^5\phi \cos^3\phi\, d\phi$ **(b)** $\int_0^{\pi/4} \tan^4 x\, dx$ **(c)** $\int \csc^4 x\, dx$ **(d)** $\int \frac{dy}{\sqrt{y^2+1}-y}$

(e) $\int \sqrt{\frac{3-x}{x-1}}\, dx$ **(f)** $\int \frac{z\, dz}{(a^2-z^2)^{5/2}}$ **(g)** $\int \frac{dp}{(a^2+p^2)^{3/2}}$ **(h)** $\int_0^a (a^2-v^2)^{3/2}\, dv$

(i) $\int \frac{x^2\, dx}{\sqrt{1-3x}}$ **(j)** $\int_0^{2a} \frac{x}{\sqrt{2ax-x^2}}\, dx$ **(k)** $\int_a^b \frac{dy}{\sqrt{(y-a)(b-y)}}$ **(l)** $\int_0^1 x^3(1-x^2)^{3/2}\, dx$

(m) $\int \frac{\cos\theta}{1+2\cos\theta}\, d\theta$ **(n)** $\int \frac{dx}{x\sqrt{x^2-1}}$ **(o)** $\int \sin^2 2\theta\, d\theta$ **(p)** $\int \sin^{-1} x\, dx$

(q) $\int v^3 \ln v\, dv$ **(r)** $\int e^{-2\theta} \sin 3\theta\, d\theta$ **(s)** $\int x\sqrt{1+x}\, dx$ **(t)** $\int u^2 e^{-\frac{1}{2}u}\, du$

(u) $\int \frac{xe^x\, dx}{(1+x)^2}$ **(v)** $\int_0^{\pi/2} \sin^3\theta(\sin^3\theta + \cos^3\theta)\, d\theta$

5. Evaluate:

(a) $\int \sec^{-1} x\, dx$ **(b)** $\int x \ln(x+1)\, dx$ **(c)** $\int \theta^2 \cos 2\theta\, d\theta$ **(d)** $\int \frac{\sin^2\theta\cos\theta}{1-\sin\theta}\, d\theta$

(e) $\int_0^{\pi/2} \sin x \cos^2 2x\, dx$ **(f)** $\int \frac{dx}{x\sqrt{1+4x^2}}$ **(g)** $\int_{\pi/2}^{\pi/4} \cot^3\theta\, d\theta$ **(h)** $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

(i) $\int \frac{x+\sin x}{1+\cos x}\, dx$ **(j)** $\int_a^b \sqrt{\frac{b-x}{x-a}}\, dx \quad (b > a)$ **(k)** $\int \frac{dx}{x^2\sqrt{\alpha x^2+\beta}}$

(l) $\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}$ **(m)** $\int \frac{x^2+5x+4}{x^4+5x^2+4}\, dx$ **(n)** $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$

(o) $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$ **(p)** $\int \frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^3}$ **(q)** $\int \frac{x\, dx}{\sqrt[4]{x^3(a-x)}}$

(r) $\int \frac{dx}{\sqrt{ax^2+bx+c}} \quad (a > 0)$ **(s)** $\int \sqrt{2+x-x^2}\, dx$ **(t)** $\int x\sqrt{x^4+2x^2-1}\, dx$

(u) $\int \frac{x^2+1}{x\sqrt{x^4+1}}\, dx$ **(v)** $\int \frac{x^2\, dx}{\sqrt{1+x-x^2}}$

6. Evaluate:

(a) $\int \frac{x\, dx}{(1+x)\sqrt{1-x-x^2}}$ **(b)** $\int \frac{dx}{x+\sqrt{x^2+x+1}}$ **(c)** $\int \frac{dx}{x-\sqrt{-3+4x-x^2}}$

(d) $\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\, dx$ **(e)** $\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})}$ **(f)** $\int xe^x \cos x\, dx$

(g) $\int xe^x \sin x\, dx$ **(h)** $\int x^2 e^x \cos x\, dx$ **(i)** $\int xe^x \sin^2 x\, dx$

(j) $\int x^2 e^x \sin^2 x\, dx$ **(k)** $\int \frac{dx}{(1+e^x)^2}$ **(l)** $\int \frac{dx}{1+e^{x/2}+e^{x/3}+e^{x/6}}$

(m) $\int\left(1-\frac{2}{x}\right)^2 e^x\,dx$ **(n)** $\int\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2 dx$ **(o)** $\int(x+|x|)^2\,dx$

(p) $\int\frac{\ln x}{(1+x^2)^{3/2}}dx$ **(q)** $\int x\tan^{-1}x\ln(1+x^2)dx$ **(r)** $\int\frac{x^3\cos^{-1}x}{\sqrt{1-x^2}}dx$

(s) $\int\frac{\cos^3 x+1}{\sin^2 x}dx$ **(t)** $\int\frac{a^x}{a^{2x}+1}dx$ **(u)** $\int e^{\sqrt{x}}dx$

(v) $\int\frac{dx}{1+\sqrt{x}+\sqrt{1+x}}, x=\left(\frac{u^2-1}{2u}\right)^2$ **(w)** $\int x|x|\ dx$ **(x)** $\int\max(1,x^2)\ dx$

(y) $\int\{|1+x|-|1-x|\}\,dx$ **(z)** $\int_{1/e}^{e}|\log_{10}x|\,dx$

7. **(i)** Evaluate $\int\frac{dx}{x^2(1+x)}$.

(ii) Show that $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$ and prove that $\int_0^{\pi/2}\frac{x\,dx}{\sin x+\cos x}=\frac{\pi\ln(1+\sqrt{2})}{2\sqrt{2}}$.

8. Prove that, if $b>a>0$, then $\int_a^b\frac{dx}{\sqrt{(x-a)(b-x)}}=\pi$, hence show that if $\alpha>\beta$, then

$$\int_\alpha^\beta\frac{d\theta}{\sqrt{(e^\theta-e^\alpha)(e^\beta-e^\theta)}}=\pi e^{-\frac{1}{2}(\alpha+\beta)}.$$

9. **(i)** Evaluate $\int_{1/x}^{x}\frac{x^2}{1+x^3}dx$, where $x>1$.

(ii) Find a substitution that transforms $\int_{1/\alpha}^{\alpha}\frac{1}{1+x^3}dx$ to $\int_{1/\alpha}^{\alpha}\frac{x}{1+x^3}dx$.

(iii) By considering the sum of the two integrals in **(ii)** in the case $\alpha=2$, evaluate: $\int_{1/2}^{2}\frac{1}{1+x^3}dx$.

(iv) Would you expect $\int_{1/\alpha}^{\beta}\frac{1}{1+x^3}dx$ and $\int_{1/\alpha}^{\beta}\frac{x}{1+x^3}dx$ to be equal when $\beta>\alpha>1$.

Justify your answer.

10. Use $\sin(a-b)=\sin[(x+a)-(x+b)]$ and $\cos(a-b)=\cos[(x+a)-(x+b)]$ to evaluate

$$\int\frac{dx}{\sin(x+a)\sin(x+b)},\quad\int\frac{dx}{\sin(x+a)\cos(x+b)},\quad\int\frac{dx}{\cos(x+a)\cos(x+b)}.$$

11. If f is continuous on $[a, b]$, show that $\int_a^b f(x)\,dx=(b-a)\int_0^1 f[a+(b-a)x]\,dx$.

12. Show that

(a) $\frac{2}{5}<\int_1^2\frac{x\,dx}{x^2+1}<\frac{1}{2}$ **(b)** $1<\int_0^1 e^{x^2}dx<e$ **(c)** $1<\int_0^{\pi/2}\frac{\sin x}{x}dx<\frac{\pi}{2}$